

# REEXAMINATION OF AN INFORMATION GEOMETRIC CONSTRUCTION OF ENTROPIC INDICATORS OF COMPLEXITY

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Information geometry and inductive inference methods can be used to model dynamical systems in terms of their probabilistic description on curved statistical manifolds.

In this article, we present a formal conceptual reexamination of the information geometric construction of entropic indicators of complexity for statistical models. Specifically, we present conceptual advances in the interpretation of the information geometric entropy (IGE), a statistical indicator of temporal complexity (chaoticity) defined on curved statistical manifolds underlying the probabilistic dynamics of physical systems.

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## I. INTRODUCTION

The mystery of the origin of life and the unfolding of its evolution is perhaps the most fascinating topic that motivates the description and, to a certain extent, the understanding of the extremely elusive concept of complexity [1–3]. From a more pragmatic point of view, its description and understanding is also motivated by the question of how complex is quantum motion. This issue is of primary importance in quantum information science, having deep connections to entanglement and decoherence. However, our knowledge of the relations between complexity, dynamical stability, and chaoticity in a fully quantum domain is not satisfactory [4, 5]. The concept of complexity is very difficult to define and its origin is not fully understood [6–8]. It is mainly for these reasons that several quantitative measures of complexity have appeared in the scientific literature [1, 2]. In classical physics, measures of complexity are understood in a better satisfactory manner. The Kolmogorov-Sinai metric entropy [9, 10], the sum of all positive Lyapunov exponents [11], is a powerful indicator of unpredictability in classical systems. It measures the algorithmic complexity of classical trajectories [12–15]. Other known measures of complexity are the logical depth [16], the thermodynamic depth [17], the computational complexity [18] and stochastic complexity [19] to name a few. For instance the logical and thermodynamic depths consider complex (roughly speaking) whatever can be reached only through a difficult path. Each one of these complexity measures captures to some degree our intuitive ideas about the meaning of complexity. Some of them just apply to computational tasks and unfortunately, only very few of them may be generalized so that their applications can be extended to actual physical processes. Ideally, a good definition of complexity should be mathematically rigorous as well as intuitive so as to allow for the analysis of complexity-related problems in computation theory and statistical physics. For obvious reasons, a quantitative measure of complexity is genuinely useful if its range of applicability is not limited to a few unrealistic applications. For similar reason, in order to properly define measures of complexity, the reasons for defining such a measure should be clearly stated as well as what feature the measure is intended to capture.

One of the major goals of physics is modeling and predicting natural phenomena by using relevant information about the system of interest. Taking this statement seriously, it is reasonable to expect that the laws of physics should reflect the methods for manipulating information. Indeed, the less controversial opposite point of view may be considered where the laws of physics are used to manipulate information. This is exactly the point of view adopted in quantum information science where information is manipulated using the laws of quantum mechanics [20]. An alternative viewpoint may be explored where laws of physics are nothing but rules of inference [21]. In this view the laws of physics are not laws of nature but merely reflect the rules we follow when processing the information that happens to be relevant to the physical problem under consideration.

Inference is the process of drawing conclusions from available information. When the information available is sufficient to make unequivocal, unique assessments of truth, we speak of making deductions: on the basis of this or that information we deduce that a certain proposition is true. In cases where we do not have statements that lead to unequivocal conclusions, we speak of using inductive reasoning and the system for this reasoning is probability

theory [22]. The word "induction" refers to the process of using limited information about a few special cases to draw conclusions about more general situations. Following this alternative line of thought, we extended the applicability of information geometric techniques [23] and inductive inference methods [24–28] to computational problems of interest in classical and quantum physics, especially with regard to complexity characterization of dynamical systems in terms of their probabilistic description on curved statistical manifolds. Moreover, we seek to identify relevant measures of chaoticity of such an information geometrodynamical approach to chaos (IGAC) [29–36].

In this article, we present a formal and conceptual reexamination of the information geometric entropy (IGE) [35], a statistical indicator of temporal complexity (chaoticity) of dynamical systems in terms of their probabilistic description using information geometry and inductive inference.

We emphasize we do not present here any new application of the IGAC (for instance, one of our most recent applications appears in [36]), however (and, most importantly) we do report some relevant conceptual advances in the interpretation of the IGE as a useful measure of complexity for statistical models suitable for probabilistic descriptions of dynamical systems.

The layout of this article is as follows. In Section II, we briefly review our information geometric approach to the description of complex systems by using information geometry and inductive inference. In Section III, we focus on the key-steps leading to the construction of the IGE and on its conceptual interpretation. Finally, in Section IV we present our final remarks.

## II. COMPLEXITY ON CURVED MANIFOLDS

IGAC [29–34] is a theoretical framework developed to study chaos in informational geodesic flows describing physical systems. The reformulation of dynamics in terms of a geodesic problem allows for the application of a wide range of well-known geometric techniques to the investigation of the solution space and properties of the equations of motion. All dynamical information is collected into a single geometric object (namely, the manifold on which geodesic flow is induced) in which all the available manifest symmetries of the system are retained. For instance, integrability of the system is connected with existence of Killing vectors and tensors on this manifold. The sensitive dependence of trajectories on initial conditions, which is a key ingredient of chaos, can be investigated by using the equation of geodesic deviation. IGAC is the information-geometric analogue of conventional geometrodynamical approaches [37, 38] where the classical configuration space  $\Gamma_E$  is replaced by a statistical manifold  $\mathcal{M}_S$  with the additional possibility of considering chaotic dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat). It is an information-geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow [39]). In the Riemannian [37] and Finslerian [38] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i.e., the sum of positive Lyapunov exponents) [40].

An  $n$ -dimensional  $C^\infty$  differentiable manifold (or more simply, a manifold) is a set of points  $\mathcal{M}$  admitting coordinate systems  $\mathcal{C}_\mathcal{M}$  and satisfies the following two conditions: 1) each element  $c \in \mathcal{C}_\mathcal{M}$  is a one-to-one mapping from  $\mathcal{M}$  to some open subset of  $\mathbb{R}^n$ ; 2) For all  $c \in \mathcal{C}_\mathcal{M}$ , given any one-to-one mapping  $\xi$  from  $\mathcal{M}$  to  $\mathbb{R}^n$ , we have that  $\xi \in \mathcal{C}_\mathcal{M} \Leftrightarrow \xi \circ c^{-1}$  is a  $C^\infty$  diffeomorphism. In this article, the points of  $\mathcal{M}$  are probability distributions. Furthermore, we consider Riemannian manifolds  $(\mathcal{M}, g)$ . The Riemannian metric  $g$  is not naturally determined by the structure of  $\mathcal{M}$  as a manifold. In principle, it is possible to consider an infinite number of Riemannian metrics on  $\mathcal{M}$ . A fundamental assumption in the information geometric framework is the choice of the Fisher-Rao information metric as the metric that underlies the Riemannian geometry of probability distributions [23, 41, 42], namely

$$g_{\mu\nu}(\Theta) \stackrel{\text{def}}{=} \int dX p(X|\Theta) \partial_\mu \log p(X|\Theta) \partial_\nu \log p(X|\Theta) = - \left( \frac{\partial^2 \mathcal{S}(\Theta', \Theta)}{\partial \Theta'^\mu \partial \Theta'^\nu} \right)_{\Theta'=\Theta}, \quad (1)$$

with  $\mu, \nu = 1, \dots, n$  for an  $n$ -dimensional manifold;  $\partial_\mu = \frac{\partial}{\partial \Theta^\mu}$  and  $\mathcal{S}(\Theta', \Theta)$  represents the logarithmic relative entropy [43],

$$\mathcal{S}(\Theta', \Theta) = - \int dX p(X|\Theta') \log \left( \frac{p(X|\Theta')}{p(X|\Theta)} \right). \quad (2)$$

The quantity  $X$  labels the microstates of the system. The choice of the information metric can be motivated in several ways, the strongest of which is Cencov's characterization theorem [44]. In this theorem, Cencov proves that the information metric is the only Riemannian metric (except for a constant scale factor) that is invariant under a family of probabilistically meaningful mappings termed congruent embeddings by Markov morphism [44, 45].

A geodesic on a  $n$ -dimensional curved statistical manifold  $\mathcal{M}_S$  represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates  $\Theta_{\text{initial}}$  and  $\Theta_{\text{final}}$ , respectively. Each point of the geodesic represents a macrostate parametrized by the macroscopic dynamical variables  $\Theta \equiv (\theta^1, \dots, \theta^n)$  defining the macrostate of the system. Each component  $\theta^k$  with  $k = 1, \dots, n$  is solution of the geodesic equation,  $\frac{d^2\theta^k}{d\tau^2} + \Gamma_{lm}^k \frac{d\theta^l}{d\tau} \frac{d\theta^m}{d\tau} = 0$ . Furthermore, each macrostate  $\Theta$  is in a one-to-one correspondence with the probability distribution  $p(X|\Theta)$ . This is a distribution of the microstates  $X$ . The set of macrostates forms the parameter space  $\mathcal{D}_\Theta$  while the set of probability distributions forms the statistical manifold  $\mathcal{M}_S$ . Applications of the IGAC using the IGE as a suitable indicator of temporal complexity (chaoticity) appear in [29–34].

### III. THE INFORMATION GEOMETRIC ENTROPY

In this Section we focus on the key-steps leading to the construction of the IGE and comment on its conceptual interpretation.

#### A. Preliminaries

Once the distances among probability distributions have been assigned using the metric tensor  $g_{\mu\nu}(\Theta)$ , a natural next step is to obtain measures for extended regions in the space of distributions. Consider an  $n$ -dimensional volume of the statistical manifold  $\mathcal{M}_s$  of distributions  $p(X|\Theta)$  labelled by parameters  $\Theta^\mu$  with  $\mu = 1, \dots, n$ . The parameters  $\Theta^\mu$  are coordinates for the point  $p$  and in these coordinates it may not be obvious how to write an expression for a volume element  $d\mathcal{V}_{\mathcal{M}_s}$ . However, within a sufficiently small region any curved space looks flat. That is to say, curved spaces are "locally flat". The idea then is rather simple: within that very small region, we should use Cartesian coordinates wherein the metric takes a very simple form, namely the identity matrix  $\delta_{\mu\nu}$ . In locally Cartesian coordinates  $\chi^\alpha$  the volume element is given by the product  $d\mathcal{V}_{\mathcal{M}_s} = d\chi^1 d\chi^2 \dots d\chi^n$ , which in terms of the old coordinates read,

$$d\mathcal{V}_{\mathcal{M}_s} = \left| \frac{\partial \chi}{\partial \Theta} \right| d\Theta^1 d\Theta^2 \dots d\Theta^n = \left| \frac{\partial \chi}{\partial \Theta} \right| d^n \Theta. \quad (3)$$

The problem at hand then is the calculation of the Jacobian  $\left| \frac{\partial \chi}{\partial \Theta} \right|$  of the transformation that takes the metric  $g_{\mu\nu}$  into its Euclidean form  $\delta_{\mu\nu}$ . Let the new coordinates be defined by  $\chi^\mu = \Xi^\mu(\Theta^1, \dots, \Theta^n)$ . A small change  $d\Theta$  corresponds to a small change  $d\chi$ ,

$$d\chi^\mu = X_m^\mu d\Theta^m \text{ where } X_m^\mu \stackrel{\text{def}}{=} \frac{\partial \chi^\mu}{\partial \Theta^m} \quad (4)$$

and the Jacobian is given by the determinant of the matrix  $X_m^\mu$ ,  $\left| \frac{\partial \chi}{\partial \Theta} \right| = |\det(X_m^\mu)|$ . The distance between two neighboring points is the same whether we compute it in terms of the old or the new coordinates,  $dl^2 = g_{\mu\nu} d\Theta^\mu d\Theta^\nu = \delta_{\alpha\beta} d\chi^\alpha d\chi^\beta$ . Therefore the relation between the old and the new metric is  $g_{\mu\nu} = \delta_{\alpha\beta} X_\mu^\alpha X_\nu^\beta$ . Taking the determinant of  $g_{\mu\nu}$ , we obtain  $g \stackrel{\text{def}}{=} \det(g_{\mu\nu}) = [\det(X_\mu^\alpha)]^2$  and therefore  $|\det(X_\mu^\alpha)| = \sqrt{g}$ . Finally, we have succeeded in expressing the volume element totally in terms of the coordinates  $\Theta$  and the known metric  $g_{\mu\nu}(\Theta)$ ,  $d\mathcal{V}_{\mathcal{M}_s} = \sqrt{g} d^n \Theta$ . Thus, the volume of any extended region on the manifold is given by,

$$\mathcal{V}_{\mathcal{M}_s} = \int d\mathcal{V}_{\mathcal{M}_s} = \int \sqrt{g} d^n \Theta. \quad (5)$$

Observe that  $\sqrt{g} d^n \Theta$  is a scalar quantity and is therefore invariant under orientation preserving general coordinate transformations  $\Theta \rightarrow \Theta'$ . The square root of the determinant  $g(\Theta)$  of the metric tensor  $g_{\mu\nu}(\Theta)$  and the flat infinitesimal volume element  $d^n \Theta$  transform as,

$$\sqrt{g(\Theta)} \xrightarrow{\Theta \rightarrow \Theta'} \left| \frac{\partial \Theta'}{\partial \Theta} \right| \sqrt{g(\Theta')}, \quad d^n \Theta \xrightarrow{\Theta \rightarrow \Theta'} \left| \frac{\partial \Theta}{\partial \Theta'} \right| d^n \Theta', \quad (6)$$

respectively. Therefore, it follows that,

$$\sqrt{g(\Theta)} d^n \Theta \xrightarrow{\Theta \rightarrow \Theta'} \sqrt{g(\Theta')} d^n \Theta'. \quad (7)$$

Equation (7) implies that the infinitesimal statistical volume element is invariant under general coordinate transformations that preserve orientation (that is, with positive Jacobian).

## B. The Formal Construction

The elements (or points)  $\{p(X|\Theta)\}$  of an  $n$ -dimensional curved statistical manifold  $\mathcal{M}_s$  are parametrized using  $n$  real valued variables  $(\theta^1, \dots, \theta^n)$ ,

$$\mathcal{M}_s \stackrel{\text{def}}{=} \left\{ p(X|\Theta) : \Theta = (\theta^1, \dots, \theta^n) \in \mathcal{D}_\Theta^{(\text{tot})} \right\}. \quad (8)$$

The set  $\mathcal{D}_\Theta^{(\text{tot})}$  is the entire parameter space (available to the system) and is a subset of  $\mathbb{R}^n$ ,

$$\mathcal{D}_\Theta^{(\text{tot})} \stackrel{\text{def}}{=} \bigotimes_{k=1}^n \mathcal{I}_{\theta^k} = (\mathcal{I}_{\theta^1} \otimes \mathcal{I}_{\theta^2} \dots \otimes \mathcal{I}_{\theta^n}) \subseteq \mathbb{R}^n \quad (9)$$

where  $\mathcal{I}_{\theta^k}$  is a subset of  $\mathbb{R}$  and represents the entire range of allowable values for the macrovariable  $\theta^k$ . For example, considering the statistical manifold of one-dimensional Gaussian probability distributions parametrized in terms of  $\Theta = (\mu, \sigma)$ , we obtain

$$\mathcal{D}_\Theta^{(\text{tot})} = \mathcal{I}_\mu \otimes \mathcal{I}_\sigma = [(-\infty, +\infty) \otimes (0, +\infty)] \subseteq \mathbb{R}^2. \quad (10)$$

In the IGAC, we are interested in a probabilistic description of the evolution of a given system in terms of its corresponding probability distribution on  $\mathcal{M}_s$  which is homeomorphic to  $\mathcal{D}_\Theta^{(\text{tot})}$ . Assume we are interested in the evolution from  $\tau_{\text{initial}}$  to  $\tau_{\text{final}}$ . Within the present probabilistic description, this equivalent to studying the shortest path (or, in terms of the ME method [24], the maximally probable path) leading from  $\Theta(\tau_{\text{initial}})$  to  $\Theta(\tau_{\text{final}})$ .

Is there a way to quantify the "complexity" of such path? We propose that the IGE  $\mathcal{S}_{\mathcal{M}_s}(\tau)$  is a good complexity quantifier [29–34]. In what follows, we highlight the key-points leading to the construction of this quantity.

We posit that a suitable indicator of temporal complexity within the IGAC framework is provided by the *information geometric entropy* (IGE)  $\mathcal{S}_{\mathcal{M}_s}(\tau)$  [32],

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \stackrel{\text{def}}{=} \log \widetilde{\text{vol}} \left[ \mathcal{D}_\Theta^{(\text{geodesic})}(\tau) \right]. \quad (11)$$

The average dynamical statistical volume  $\widetilde{\text{vol}} \left[ \mathcal{D}_\Theta^{(\text{geodesic})}(\tau) \right]$  is defined as,

$$\widetilde{\text{vol}} \left[ \mathcal{D}_\Theta^{(\text{geodesic})}(\tau) \right] \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \infty} \left( \frac{1}{\tau} \int_0^\tau d\tau' \text{vol} \left[ \mathcal{D}_\Theta^{(\text{geodesic})}(\tau') \right] \right), \quad (12)$$

where the "tilde" symbol denotes the operation of temporal average. For the sake of clarity, we point out that in the RHS of (12), we intend to preserve the temporal-dependence by considering the asymptotic leading term in the limit of  $\tau$  approaching infinity. The volume  $\text{vol} \left[ \mathcal{D}_\Theta^{(\text{geodesic})}(\tau') \right]$  is given by,

$$\text{vol} \left[ \mathcal{D}_\Theta^{(\text{geodesic})}(\tau') \right] \stackrel{\text{def}}{=} \int_{\mathcal{D}_\Theta^{(\text{geodesic})}(\tau')} \rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n) d^n \Theta, \quad (13)$$

where  $\rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n)$  is the so-called Fisher density and equals the square root of the determinant of the metric tensor  $g_{\mu\nu}(\Theta)$ ,

$$\rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n) \stackrel{\text{def}}{=} \sqrt{g((\theta^1, \dots, \theta^n))}. \quad (14)$$

The integration space  $\mathcal{D}_\Theta^{(\text{geodesic})}(\tau')$  in (13) is defined as follows,

$$\mathcal{D}_\Theta^{(\text{geodesic})}(\tau') \stackrel{\text{def}}{=} \left\{ \Theta \equiv (\theta^1, \dots, \theta^n) : \theta^k(0) \leq \theta^k \leq \theta^k(\tau') \right\}, \quad (15)$$

where  $k = 1, \dots, n$  and  $\theta^k \equiv \theta^k(s)$  with  $0 \leq s \leq \tau'$  such that,

$$\frac{d^2 \theta^k(s)}{ds^2} + \Gamma_{lm}^k \frac{d\theta^l}{ds} \frac{d\theta^m}{ds} = 0. \quad (16)$$

The integration space  $\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')$  in (15) is a  $n$ -dimensional subspace of the whole (permitted) parameter space  $\mathcal{D}_{\Theta}^{(\text{tot})}$ . The elements of  $\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')$  are the  $n$ -dimensional macrovariables  $\{\Theta\}$  whose components  $\theta^k$  are bounded by specified limits of integration  $\theta^k(0)$  and  $\theta^k(\tau')$  with  $k = 1, \dots, n$ . The limits of integration are obtained via integration of the  $n$ -dimensional set of coupled nonlinear second order ordinary differential equations characterizing the geodesic equations. Formally, the IGE  $\mathcal{S}_{\mathcal{M}_s}(\tau)$  is defined in terms of a averaged parametric  $(n+1)$ -fold integral ( $\tau$  is the parameter) over the multidimensional geodesic paths connecting  $\Theta(0)$  to  $\Theta(\tau)$ .

In conventional approaches to chaoticity, chaos is specified within the context of dynamical systems themselves. The existence of classical dynamical chaos can be inferred from the exponential divergence of the Jacobi vector field associated to the geodesic flow which coincides with the natural microscopic dynamics, that is the dynamics described by Newton's equation of motion. Furthermore, dynamical chaos requires two basic ingredients: stretching and folding of phase space trajectories. In geometric language, chaos requires hyperbolicity and compactness of the manifold where a geodesic flow "lives".

In our information geometric approach to chaos, chaoticity is specified within the context of suitable statistical manifolds underlying the probabilistic (entropic) dynamics of dynamical systems when only incomplete information on the systems is available. Indeed, our approach could have a wide range of applicability. For instance, as a special limiting case, Newtonian dynamics can be derived from prior information codified into an appropriate statistical model [46]. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the information metric. The trajectory (geodesic) follows from a principle of inference, the method of Maximum Entropy. No additional "physical" postulates such as an equation of motion, or an action principle, nor the concepts of momentum and of phase space, not even the notion of time, need to be postulated.

A geodesic on a curved statistical manifold represents the maximum probability path a dynamical system explores in its (probabilistic and statistical) evolution between the initial and the final macrostates on the statistical manifold. Each point of the geodesic is parametrized by the macroscopic dynamical variables defining the macrostate of the system. Furthermore, each macrostate is in a one-to-one relation with the probability distribution representing the maximally probable description of the system being considered. The set of macrostates forms the parameter space while the set of probability distributions forms the statistical manifold. The parameter space is homeomorphic to the statistical manifold. The resulting entropic dynamics reproduces the Newtonian dynamics of any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

Once again, we point out that several interesting applications of the IGE appear in the literature [29–36]. For instance in [29], we proposed a novel information-geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain in a tilted (transverse) external magnetic field and conjectured our findings might find some potential physical applications in quantum energy level statistics. However, in the next Section we will report our latest conceptual advances in the interpretation of the IGE.

### C. The Conceptual Interpretation

The quantity  $\text{vol}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')]$  is the volume of the effective parameter space explored by the system at time  $\tau'$ . Its faithful geometric visualization may be highly non trivial, especially in high-dimensional spaces [47]. The temporal average has been introduced in order to average out the possibly very complex fine details of the entropic dynamical description of the system on  $\mathcal{M}_S$  [48]. Thus, we provide a coarse-grained-like (or randomized-like) inferential description of the system chaotic dynamics. The long-term asymptotic temporal behavior is adopted in order to properly characterize dynamical indicators of chaoticity (for instance, Lyapunov exponents, entropies, etc.) eliminating the effects of transient effects which enters the computation of the expected value of  $\text{vol}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')]$ . In chaotic transients, one observes that typical initial conditions behave in an apparently chaotic manner for a possibly long time, but then asymptotically approach a nonchaotic attractor in a rapid fashion. We term the asymptotic quantity  $\widetilde{\text{vol}}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau)]$ ,

$$\widetilde{\text{vol}}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau)] \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \infty} \left( \frac{1}{\tau} \int_0^{\tau} \left[ \int_{\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')} \rho_{(\mathcal{M}_s, g)}(\theta^1, \dots, \theta^n) d^n \vec{\theta} \right] d\tau' \right) = \exp(\mathcal{S}_{\mathcal{M}_s}(\tau)), \quad (17)$$

the *information geometric complexity* of the geodesic paths on  $\mathcal{M}_S$ . Again, we emphasize that in the RHS of (17), we intend to preserve the temporal-dependence by considering the asymptotic leading term in the limit of  $\tau$  approaching infinity. To have a more intuitive understanding of  $\widetilde{\text{vol}}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau)]$ , we recall that in going from  $\Theta(\tau_0) \stackrel{\text{def}}{=} \Theta_{\text{initial}}$

to  $\Theta(\tau_f) \stackrel{\text{def}}{=} \Theta_{\text{final}}$  we assume the system passes through a continuous succession of (infinitesimally close) intermediate steps. For instance at the  $\bar{k}$ -th step (with  $\bar{k} = 1, 2, 3, \dots$ ), we study the evolution of the system from  $\Theta(\tau_{\bar{k}-1})$  at  $\tau_{\bar{k}-1}$  to  $\Theta(\tau_{\bar{k}})$  at  $\tau_{\bar{k}} = \tau_{\bar{k}-1} + d\tau$ . At the  $(\bar{k} + 1)$ -th step, we study the evolution of the system from  $\Theta(\tau_{\bar{k}})$  at  $\tau_{\bar{k}}$  to  $\Theta(\tau_{\bar{k}+1})$  at  $\tau_{\bar{k}+1} = \tau_{\bar{k}} + d\tau$ , and so on. Now, let us consider the following adimensional quantity characterizing two consecutive steps, the  $\bar{k}$ -th and  $(\bar{k} + 1)$ -th steps,

$$\left( \frac{\delta \widetilde{vol}}{\widetilde{vol}} \right)_{\bar{k} \rightarrow \bar{k}+1} \stackrel{\text{def}}{=} \frac{\widetilde{vol}_{[\tau_{\bar{k}}, \tau_{\bar{k}+1}]} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right] - \widetilde{vol}_{[\tau_{\bar{k}-1}, \tau_{\bar{k}}]} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right]}{\widetilde{vol}_{[\tau_{\bar{k}-1}, \tau_{\bar{k}}]} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right]} \quad (18)$$

where the average infinitesimal statistical volume explored from  $\tau_m$  to  $\tau_M$ , i.e.,  $\widetilde{vol}_{[\tau_m, \tau_M]} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right]$ , is given by,

$$\widetilde{vol}_{[\tau_m, \tau_M]} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right] \stackrel{\text{def}}{=} \frac{1}{\tau_M - \tau_m} \int_{\tau_m}^{\tau_M} \widetilde{vol} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right] d\tau'. \quad (19)$$

The quantity  $\left( \frac{\delta \widetilde{vol}}{\widetilde{vol}} \right)_{\bar{k} \rightarrow \bar{k}+1}$  is the average relative increment of the volume of the statistical macrospace explored by the system in its dynamical evolution between two infinitesimally close and consecutive steps (macrostates). The temporal behavior of  $\left( \frac{\delta \widetilde{vol}}{\widetilde{vol}} \right)_{\bar{k} \rightarrow \bar{k}+1}$  is a rough indicator of the presence of complex behavior in the evolution being considered. To have a more reliable complexity indicator, a step-average over an asymptotically infinite number  $N$  of steps would be required. Therefore, the quantity to consider becomes  $\frac{\delta \widetilde{vol}}{\widetilde{vol}}$ ,

$$\frac{\delta \widetilde{vol}}{\widetilde{vol}} \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \left( \frac{\delta \widetilde{vol}}{\widetilde{vol}} \right)_{\bar{k} \rightarrow \bar{k}+1} \quad (20)$$

where  $\left( \frac{\delta \widetilde{vol}}{\widetilde{vol}} \right)_{\bar{k} \rightarrow \bar{k}+1}$  defined in (18). For instance, a linear increase of  $\frac{\delta \widetilde{vol}}{\widetilde{vol}}$  would be a reasonable manifestation of the presence of chaoticity (temporal complexity, dynamical stochasticity),

$$\frac{\delta \widetilde{vol}}{\widetilde{vol}} \stackrel{N, \tau \rightarrow \infty}{\approx} \mathcal{K}_{IG} \tau \Leftrightarrow \widetilde{vol} \stackrel{N, \tau \rightarrow \infty}{\approx} \exp(\mathcal{K}_{IG} \tau). \quad (21)$$

The quantity  $\mathcal{K}_{IG} \stackrel{\tau \rightarrow \infty}{\approx} \frac{d\mathcal{S}_{\mathcal{M}_S}(\tau)}{d\tau}$  is a model parameter of the complex system and depends on the temporal evolution of the statistical macrovariables [29]. It may be interpreted as playing a role similar to that of the KS entropy rate (sum of all positive Lyapunov exponents of the dynamical trajectories) and it is, in principle, an experimentally observable quantity [29]. We emphasize there may be physical processes described by several characteristic time scales where the exponential divergence of  $\widetilde{vol}$  may not be required, although in the presence of chaoticity [49].

We point out that our construction and interpretation has some similarities with the logical and thermodynamic depths. For instance, we recall that the logical depth [16] is considered to be one of the best candidates as a measure of (statistical) complexity [1]. It is an example of a statistical complexity where the correlated structure of the systems' constituents play a key role in determining the complex path connecting the initial and final states of the system under investigation. It is a time measure of complexity and represents the run time required by a universal Turing machine executing the minimal program to reproduce a given pattern. We emphasize that such run time is obtained by a suitable averaging procedure over the various programs that will accomplish the task by weighting shorter programs more heavily. Therefore, the logical depth of any system is defined if a suitably coarse-grained description of it is encoded into a bit string. In our construction of the IGE, the temporal average has been introduced in order to average out the possibly very complex fine details of the entropic dynamical description of the system on  $\mathcal{M}_S$ . Therefore, we also provide a coarse-grained-like inferential description of the system's chaotic dynamics. Furthermore, we point out that one key objection to the thermodynamic depth [17] cannot emerge in our construction of complexity measure. The thermodynamic depth of a process is a structural measure of complexity and it represents the difference between the system's coarse- and fine-grained entropy. The "depth" of a macrostate reached by a particular trajectory is  $-\text{const} \log p_j$  (here  $\text{const} = k_B$  is chosen to be Boltzmann's constant for systems whose successive configurations can be described in the physical space of statistical mechanics), where  $p_j$  is the probability of the  $j$ -th trajectory. For the whole range of possible trajectories, the resulting weighted average is  $-\text{const} \sum_j p_j \log p_j$ . The set  $\{p_j\}$  represent probabilities which are consistent with all the measurements that have been made on the system during its history.

This way of reasoning seems very close in spirit to our complexity measure construction. However, the key-objection to the thermodynamic depth is the arbitrariness and lack of explanation of how the macrostates of the system leading to the formation of the path-trajectory are selected [50]. Instead, in our construction the selection is as objective as possible since it relies on the universal ME updating method [24] where we maximize the logarithmic relative entropy  $\mathcal{S}(\Theta_{\bar{k}-1}, \Theta_{\bar{k}})$  [43] between each pair  $(\Theta_{\bar{k}-1}, \Theta_{\bar{k}})$  of consecutive macrostates forming the path connecting the initial  $\Theta_{\text{initial}}$  to the final  $\Theta_{\text{final}}$  macrostate. The ME method of determining macroscopic paths makes no mention of randomness or other incalculable quantities. It simply chooses the distribution (macrostate) with the maximum entropy allowed by the information constraints. Thus, it selects the most uninformative distribution of microstates possible. If we chose a probability distribution with lower entropy then we would assume information we do not possess; to choose one with a higher entropy would violate the constraints of the information we do possess. Thus the maximum entropy distribution is the only reasonable distribution. Are there other methods of updating? Yes, but the ME method is the most fundamental, following the rules of probability theory as outlined by Cox. He proved that probability theory is the only logically consistent theory of inductive inference [22].

#### IV. FINAL REMARKS

In this article, we have presented a formal and conceptual reexamination of the information geometric entropy (IGE), a statistical indicator of temporal complexity (chaoticity) of dynamical systems in terms of their probabilistic description on curved statistical manifolds.

In our information geometric approach, the information geometric complexity represents a statistical measure of complexity of the macroscopic path  $\Theta \stackrel{\text{def}}{=} \Theta(\tau)$  on  $\mathcal{M}_S$  connecting the initial and final macrostates  $\Theta_i$  and  $\Theta_f$ , respectively. The path  $\Theta(\tau)$  is obtained via integration of the geodesic equations on  $\mathcal{M}_S$  generated by the universal ME updating method. At a discrete level, the path  $\Theta(\tau)$  can be described in terms of an infinite continuous sequence of intermediate macroscopic states,  $\Theta(\tau) = [\Theta_i, \dots, \Theta_{\bar{k}-1}, \Theta_{\bar{k}}, \Theta_{\bar{k}+1}, \dots, \Theta_f]$  with  $\Theta_j = \Theta(\tau_j)$ , determined via the logarithmic relative entropy maximization procedure subjected to well-specified normalization and information constraints. The nature of such constraints define the (correlated) structure of the underlying probability distribution on the particular manifold  $\mathcal{M}_S$ . In other words, the correlated structure that may emerge from our information-geometric statistical models has its origin in the valuable information about the microscopic degrees of freedom of the actual physical systems. It emerges in the ME maximization procedure via integration of the geodesic equations on  $\mathcal{M}_S$  and is finally quantified in terms of the intuitive notion of volume growth via the information geometric complexity or, in entropic terms by the IGE. The information geometric complexity is then interpreted as the volume of the statistical macrospace explored in the asymptotic limit by the system in its evolution from  $\Theta_i$  to  $\Theta_f$ . Otherwise, upon a suitable normalization procedure that renders the information geometric complexity an adimensional quantity, it represents the number of accessible macrostates (with coordinates living in the accessible parameter space) explored by the system in its evolution from  $\Theta_i$  to  $\Theta_f$ .

In our view, this work constitutes a non-trivial effort toward an understanding of the concept of complexity in dynamical systems in terms of their probabilistic description on curved statistical manifolds through information geometry and inductive inference. It is also our view that this effort should be extended to a full quantum domain [29, 30].

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